MATH 579: Combinatorics
Exam 1 Solutions

1. How many four-letter words, drawn from the usual 26 letters, contain exactly three different letters?
If there are three different letters, then one letter must be used twice (and the others used once). There are 26 ways to choose the letter used twice, and $\binom{4}{2} = 6$ ways to place that repeated letter. There are 25 ways to fill the first unused space with a different letter, and 24 ways to fill the last space with a different letter. Hence the answer is $26 \cdot 6 \cdot 25 \cdot 24 = 93600$.

2. Calculate $S(6,3)$.
The quickest way is using the explicit formula $S(6,3) = \frac{1}{3!}\sum_{j=0}^{3}(-1)^{3-j}\binom{3}{j}j^6 = \frac{1}{6}(-\binom{3}{0}0^6 + \binom{3}{1}1^6 - \binom{3}{2}2^6 + \binom{3}{3}3^6) = \frac{1}{6}(-0+3-192+729) = \frac{540}{6} = 90$.

3. How many solutions are there to $x_1 + x_2 + x_3 = 30$ in nonnegative integers, with x_1, x_2, x_3 distinct?
There are $\left(\!\binom{3}{30}\!\right) = 496$ solutions, ignoring the "distinct" restriction. For any k with $0 \le k \le 15$, we can set $x_1 = x_2 = k$, and have a unique choice $x_3 = 30 - 2k$ to solve the equation. Hence there are 16 "bad" solutions where $x_1 = x_2$. Similarly, there are 16 "bad" solutions where $x_1 = x_3$, and 16 "bad" solutions where $x_2 = x_3$. But be careful! If we compute $496 - 3 \cdot 16$, we subtract the solution $x_1 = x_2 = x_3 = 10$ three times. We need to only subtract it once, so the correct answer is $496 - 3 \cdot 16 + 2 = 450$.

4. Find a closed formula for $S(n, n-2)$, for all $n \ge 3$. Be sure to prove your answer.
The "typical" part will have a single element from $[n]$; however there are two "extra" elements. With the "extras" together, then one part will have three elements, and the rest will be singletons. There are $\binom{n}{3}$ ways to select the part with three elements. The other case has two parts of size two, and the rest as singletons. Be careful! If we count as $\binom{n}{2}\binom{n-2}{2}$, we are double-counting $\{a,b\},\{c,d\}$ as different from $\{c,d\},\{a,b\}$. We need to divide by 2! to correct this. Combining with the first case, we get $\binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{5n^3-12n^2+7n}{12} = S(n, n-2)$.

5. Let $p_k(n)$ denote the number of partitions of n into k parts, and let $t \in \mathbb{N}$. Prove that $\lim_{n\to\infty} p_{n-t}(n)$ exists, and find this limit.
For n sufficiently large (as it happens, $n \ge 2t$), we will prove that $p_{n-t}(n) = p(t)$. Take any partition of n into $n-t$ parts. Remove one from each part. This will be a partition of $n-(n-t) = t$ into at most $n-t$, possibly empty, parts. So long as $n - t \ge t$, this is exactly a partition of t into any number of parts. This is reversible; given a partition of t, we extend to have $n-t$ (possibly empty) parts, then add one to each part. We now have $n-t$ nonempty parts, and a total of $n-t+t = n$. Since $p(t)$ is independent of n, this is equal to the desired limit.

6. Count the number of functions $f : A \to B$, with $|A| = a$, $|B| = b$, with the elements of A identical to each other, and the elements of B identical to each other.
This is one entry of the Twelvefold Way. A function that we're counting is a partition of a into at most b parts. If exactly b parts, there are $p_b(a)$ such functions. If exactly $b-1$ parts, then there are $p_{b-1}(a)$ such functions. There has to be at least one part. Putting it all together, there are $p_1(a) + p_2(a) + \cdots + p_b(a)$ such functions.